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Classical solutions of a nonlinear field equation

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Abstract. The field equation for a complex scalar field with a cubic nonlinear term is studied in four-dimensional Minkowski space. Two classes of exact solutions corresponding to plane and spherical waves are presented.

1. Introduction

For a long time the classical solutions of linear (free) field equations have formed the basis for the corresponding quantum field theory, even for nonlinear field equations, where the superposition principle of solutions does not hold. In recent years this attitude has changed. Special types of solution, which are intimately connected with the nonlinearity of the field equation, have been the focus of interest. There are also new attempts to quantise nonlinear field equations starting directly from the corresponding classical solutions. It seems therefore appropriate to learn as much as possible about classical solutions of nonlinear model field theories.

In the following sections we shall consider one of the simplest models: a scalar field with quartic self-coupling in the Lagrangian. Because of its simplicity this model is of interest by itself; in addition, the corresponding nonlinear equation has also found interest in connection with gauge field theory. It is not the scope of this paper to look for all possible applications or interpretations. Instead we shall adopt a very simple-minded approach: considering the field equation as a wave equation, we shall require some symmetry properties of the solutions (as one does, for instance, in Maxwell's theory). It turns out that at least for two classes of solutions the symmetry requirement fixes the classical solutions completely. The magnitude of the solutions can be given explicitly in terms of well-known functions, whereas the phase is determined by integrals which can be evaluated only in special cases. The manifold of solutions obtained is parametrised by certain constants which are connected with boundary values for the wave field.

2. Field equation and physical quantities

The classical field to be studied here is a complex, scalar field $A(x)$ in Minkowski space:

$$x^{\mu} = (x^0, \boldsymbol{x}), \qquad x^2 = x_{\mu}x^{\mu} = (x^0)^2 - \boldsymbol{x} \cdot \boldsymbol{x}.$$

The field is to be determined from the nonlinear equation

$$\Box A - \lambda A A^* A = 0. \tag{1}$$

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We shall consider both signs of the (real) coupling constant λ. The Lagrangian density is

$$L(x)=\tfrac{1}{2}(\partial_\mu A^*)(\partial^\mu A)+\tfrac{1}{4}\lambda\,(A^*A)^2. \tag{2}$$

Other quantities of interest are the energy density

$$H(x)=\tfrac{1}{2}[|\partial^0 A|^2+|\nabla A|^2-\tfrac{1}{2}\lambda\,(A^*A)^2], \tag{3}$$

the momentum density

$$\boldsymbol{P}(x)=-\tfrac{1}{2}[(\partial^0 A^*)(\nabla A)+(\partial^0 A)(\nabla A^*)], \tag{4}$$

and the current density

$$j^\mu(x)=\mathrm{i}^{-1}(A^*\,\partial^\mu A-A\,\partial^\mu A^*). \tag{5}$$

The energy density is positive definite for $\lambda<0$. We shall observe, however, that there are also solutions of equation (1) which yield H positive definite also for $\lambda>0$.

As a consequence of equation (1) we note the local conservation laws

$$\partial_\mu j^\mu=0 \tag{6}$$

and

$$\partial^0 H+\nabla\cdot\boldsymbol{P}=0. \tag{7}$$

There are, in fact, a number of similar equations for other densities which will not be considered here. All these equations are connected with transformations which leave equation (1) invariant. This is the case for translations, Lorentz transformations, dilatations, conformal transformations and phase transformations $A\to e^{i\alpha}A$. Because of the latter invariance A is only fixed up to a constant phase factor. If we write A in the form

$$A=R(x)\,e^{i\phi(x)},$$

with real R and ϕ, we can restrict the discussion to positive values of R. A solution with constant ϕ has $j^\mu=0$ and will be called a real field. With respect to Lorentz transformations we shall consider only invariant solutions, since we want A to be a Lorentz scalar. The other invariances mean that, with any solution $A(x)$, $A(x+b)$,

$$A'(x)=\eta A(\eta x) \tag{8}$$

and

$$A''(x)=(1/\sigma(x,b))A(y), \tag{9}$$

with

$$y^\mu=(x^2/\sigma(x,b))(x^\mu/x^2+b^\mu),\qquad \sigma(x,b)=1+2bx+b^2x^2 \tag{10}$$

are also solutions of equation (1). Here η is an arbitrary real constant, and b^μ is an arbitrary real four-vector. In general we cannot expect that the solutions of equation (1) are also invariant, i.e. $A'(x)=A(x)$, $A''(x)-A(x)$. There are, however, some invariant solutions.

We shall now determine all solutions of equation (1) which correspond, in analogy to linear wave equations, to certain wave types, i.e. (*a*) plane waves and (*b*) spherical waves.

3. Plane waves

The first type of solution to be considered is one for which A depends on x only through

$$p_\mu x^\mu = p_0 x_0 - \boldsymbol{p} \cdot \boldsymbol{x},$$

with a fixed vector p^μ. Fields of this type correspond to waves whose fronts are planes perpendicular to $\boldsymbol{p}$ and propagate in the direction of this vector. Of course the most general plane wave would also contain signals propagating in the opposite direction, i.e. A should also depend on $p_0 x_0 + \boldsymbol{p} \cdot \boldsymbol{x}$. For simplicity, however, we shall consider only one coordinate here. Then we must have $p^2 \neq 0$ (otherwise there is no non-trivial solution). We shall use the abbreviations

$$g = \lambda / p^2, \qquad \tau = \sqrt{|g|}\, p_\mu x^\mu, \qquad \epsilon = \operatorname{sgn} g, \tag{11}$$

and write

$$A = R(\tau)\, e^{i\phi(\tau)}. \tag{12}$$

Denoting the derivative with respect to τ by a dot, we obtain from equation (6)

$$d(R^2 \dot{\phi})/d\tau = 0. \tag{13}$$

Thus we have

$$\dot{\phi} = D/2R^2, \tag{14}$$

with a real integration constant D. The differential equation for the radial part is obtained from equation (1) and reads

$$\ddot{R} - \epsilon R^3 - D^2/4R^2 = 0.$$

We note immediately that, with any solution $R(\tau)$, $\phi(\tau)$, $R(\tau+M)$, $\phi(\tau, M)$ is also a solution, where M is an arbitrary constant. We can use this freedom to fix the initial conditions at a special value of τ, e.g. $\tau = 0$.

The differential equation for R can be integrated once to give

$$\dot{R}^2 - \tfrac{1}{2}\epsilon R^4 + D^2/4R^2 + C = 0,$$

with another real integration constant C. Introducing

$$\rho = R^2, \tag{15}$$

the differential equation becomes

$$\dot{\rho}^2 = -D^2 - 4C\rho + 2\epsilon\rho^3.$$

Thus we have

$$\sqrt{2}\, \tau = \int_{\rho_0}^{\rho} \frac{dx}{\sqrt{f(x)}}, \tag{16}$$

where

$$f(x) = \epsilon x^3 - 2Cx - D^2/2 \tag{17}$$

and $\rho_0 = \rho(0)$. The integral on the RHS of equation (16) is an elliptic integral: thus ρ is an elliptic function. The explicit form depends on the situation of the roots of the cubic form (17) which can be written in terms of C, D and ϵ. The possibilities can be exhibited graphically. Since x corresponds to R^2 and τ is real, only the regions with $f(x)>0$ for positive x are physically relevant. For $D=0$ (real field) we have three typical cases I, II and III, shown in figures 1(a), (b) and (c) respectively.

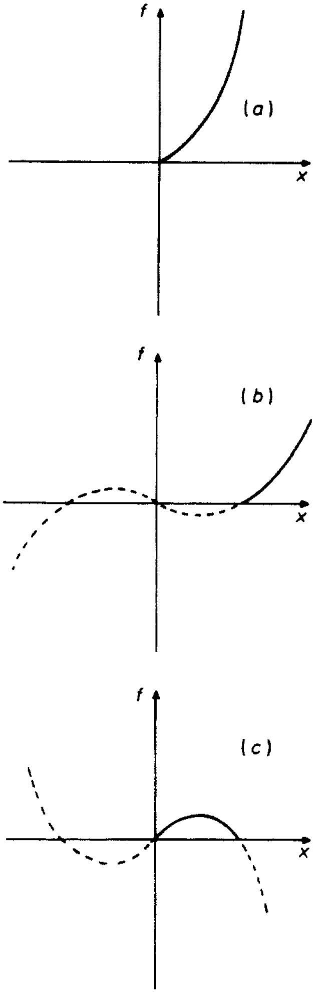

Figure 1. (a) Case I: $\epsilon=1$, $C\leqslant 0$, one real root. (b) Case II: $\epsilon=1$, $C>0$, three real roots. (c) $\epsilon=-1$, $C<0$, three real roots.

For $\epsilon=-1$ and $C>0$ we have $f<0$ for positive x; therefore this case must be excluded. These three cases are also relevant in the general case ($D\neq 0$). For case III, however, D^2 must be small enough to guarantee the existence of a region with positive f. Since D is proportional to the charge carried by the field (cf equation (19)), case III corresponds to solutions for which the charge is physically limited.

We shall express the solutions in terms of Jacobian elliptic functions $\mathrm{sn}(w/k)$, cn, dn and the elliptic integral $F(k/\psi)$. The argument $w(\tau)$ (which is linear in τ), the modulus k ($0\leqslant k\leqslant 1$) and the angle ψ are given in terms of some constants which will be specified in terms of the initial data below. The solutions in the three cases of physical relevance read as follows. If the cubic form has two complex roots and one real root $a_1 \geqslant 0$ and $\epsilon = 1$ we have

$$\begin{aligned} &\rho = a_1 + B\,\mathrm{cn}^2(w_1|k_1)/\mathrm{sn}^2(w_1|k_1)\,\mathrm{dn}^2(w_1|k_1), \\ &w_1 = (B/2)^{1/2}\tau - \tfrac{1}{2}F(k_1|\psi_1), \\ &k_1^2 = \tfrac{1}{2}(1 - 3G/2B), \qquad \cos\psi_1 = (\rho_0 - a_1 - B)/(\rho_0 - a_1 + B). \end{aligned} \tag{I}$$

For three real roots $a_2 > b_1 > b_2$ and $\epsilon = 1$ we have

$$\begin{aligned} &\rho = a_2 + (a_2 - b_2)\,\mathrm{cn}^2(w_2|k)/\mathrm{sn}^2(w_2|k), \\ &w_2 = [(a_2 - b_2)/2]^{1/2}\tau - F(k|\psi_2), \\ &k^2 = (b_1 - b_2)/(a_2 - b_2), \qquad \sin\psi_2 = [(a_2 - b_2)/(\rho_0 - b_2)]^{1/2}. \end{aligned} \tag{II}$$

For three real roots and $\epsilon = -1$ we have

$$\begin{aligned} &\rho = a_2 - (a_2 - b_1)\,\mathrm{sn}^2(w_3|k'), \\ &w_3 = [(a_2 - b_2)/2]^{1/2}\tau - F(k'|\psi_3), \\ &k'^2 = 1 - k^2 = (a_2 - b_1)/(a_2 - b_2), \qquad \sin\psi_3 = [(a_2 - \rho_0)/(a_2 - b_1)]^{1/2}. \end{aligned} \tag{III}$$

All three forms are periodic functions. It has to be noted, however, that only solution (III) is bounded, whereas the others assume infinite values within each period. If two or all three roots coincide, the solutions degenerate into elementary functions, which will be discussed below.

Once ρ is determined, we should compute ϕ from

$$\phi = \frac{D}{2}\int_0^\tau \frac{\mathrm{d}x}{\rho(x)} = \frac{D}{2\sqrt{2}}\int_{\rho_0}^{\rho} \frac{\mathrm{d}x}{x\sqrt{f(x)}}. \tag{18}$$

The first form gives $\phi = \phi(\tau)$, the second $\phi = \phi(\rho)$. Since these integrals are not at all simple, we shall write down the phase only in some special cases.

The other quantities of interest read

$$\begin{aligned} &j^\mu = \sqrt{|g|}\,p^\mu D, \\ &H = \tfrac{1}{2}|g|[(p^{02} + \boldsymbol{p}^2)|\dot{R} + \mathrm{i}R\dot{\phi}|^2 - \tfrac{1}{2}\epsilon p^2 R^4] = \tfrac{1}{2}|g|[(\boldsymbol{p}^2/2\rho)(\dot{\rho}^2 + D^2) - Cp^2], \\ &\boldsymbol{P} = |g|p^0\boldsymbol{p}|\dot{R} + \mathrm{i}R\dot{\phi}|^2 = -|g|p^0\boldsymbol{p}(\tfrac{1}{2}\epsilon\rho^2 - C), \\ &L = \tfrac{1}{2}|g|p^2(|\dot{R} + \mathrm{i}R\dot{\phi}|^2 + \tfrac{1}{2}\epsilon R^4) = \tfrac{1}{2}|g|p^2[(\dot{\rho}^2 + D^2)/2\rho + C]. \end{aligned} \tag{19}$$

Positivity properties may be read off directly. Thus we see that H is not only positive for $\lambda > 0$, but also for $\lambda > 0$ and $Cp^2 \leqslant 0$. It is perhaps interesting to observe that the expressions (19) can be integrated over τ. This is accomplished by partial integrations and use of the differential equations for R and ϕ. The result is

$$\begin{aligned} &\int^\tau H\,\mathrm{d}\tau = \tfrac{1}{6}|g|[2\boldsymbol{p}^2 R\dot{R} - C(3p^{02} + \boldsymbol{p}^2)\tau], \\ &\int^\tau \boldsymbol{P}\,\mathrm{d}\tau = \tfrac{1}{3}|g|p^0\boldsymbol{p}(R\dot{R} - 2C\tau) \\ &\int^\tau L\,\mathrm{d}\tau = \tfrac{1}{6}|g|p^2(2R\dot{R} - C\tau). \end{aligned} \tag{20}$$

These formulae are useful if one wants to compute the energy or momentum contained in a finite volume, or the corresponding contribution to the action integral (for an infinite interval one may obtain an infinite result, as for linear plane waves). It is observed that the essential dependence on τ is through ρ.

Now we shall specify the constants appearing in the solutions (I)–(III). We recall that the integration constants read

$$D = 2\rho_0\dot{\phi}_0, \qquad C = \rho_0(\tfrac{1}{2}\epsilon\rho_0 - (\dot{\rho}_0/2\rho_0)^2 - \dot{\phi}_0^2), \tag{21}$$

in terms of the initial values (the index 0 means $\tau = 0$). The solution has the form (I) for

$$-\infty \leqslant C \leqslant \tfrac{3}{4}D^{2/3}, \tag{22}$$

and we have

$$\begin{aligned} &a_1 = (D^2/4 + \sqrt{\Delta})^{1/3} + (D^2/4 - \sqrt{\Delta})^{1/3}, \qquad \Delta = D^4/16 - 8C^3\epsilon/27, \\ &G = a_1, \qquad B = (3a_1^2 - 2C)^{1/2}. \end{aligned} \tag{23}$$

H is positive for $C < 0$, but not always for $C > 0$ (unless $\lambda < 0$). Form (II) or (III) is obtained for

$$\epsilon C \geqslant \tfrac{3}{4}D^{2/3}. \tag{24}$$

The roots read

$$\begin{aligned} &a_2 = 2(2\epsilon C/3)^{1/2}\cos\alpha/3, \qquad b_{1,2} = -2(2\epsilon C/3)^{1/2}\cos(\alpha \pm \pi)/3, \\ &\cos\alpha = (3D^2/8C)(2\epsilon C/3)^{-1/2}. \end{aligned} \tag{25}$$

H is not positive definite if there are three real roots and $\lambda > 0$.

There are some special cases in which the elliptic functions are in fact elementary functions. These elementary solutions can be found most easily from the original differential equation for R, and correspond to coinciding roots of the form (17). For $\epsilon = 1$, $\Delta = 0$ and $D \neq 0$ we obtain from forms (I) or (II)

$$\begin{aligned} &\rho = a[1 + \tfrac{2}{3}\cot^2(\tau\sqrt{3a}/2 - \beta)], \\ &a = (2D^2)^{1/3}, \qquad \sin\beta = [3a/(2\rho_0 + a)]^{1/2}. \end{aligned} \tag{26}$$

H is only positive for sufficiently large argument of the cot. For $\epsilon = -1$, $\Delta = 0$ and $D \neq 0$ we obtain from form (III)

$$\rho = \rho_0 = \tfrac{1}{2}(2D^2)^{1/3}. \qquad \phi = \sqrt{\rho_0}\,\tau, \tag{27}$$

which corresponds to a monochromatic plane wave. For this solution we have $H \geqslant 0$.

Finally we shall give an exhaustive list of solutions with real fields. Then $D = 0$ and we have only one essential parameter C. The three forms given above can then be simplified using relations between Jacobian functions. The modulus of the elliptic functions becomes

$$k_1^2 = k^2 = k'^2 = \tfrac{1}{2}. \tag{28}$$

With the abbreviation

$$a = (2C)^{1/2}, \tag{29}$$

the argument can be written

$$w_i = a^{1/2}\tau - F_i, \qquad i = 1, 2, 3, \tag{30}$$

and the solutions are

$$\epsilon = +1, \qquad C < 0,$$

$$R = (2a)^{1/2} \frac{\mathrm{dn}(w_1)}{\mathrm{sn}(w_1)\,\mathrm{cn}(w_1)} = 2a^{1/2}\left(\frac{1+\mathrm{dn}(2w_1)}{1-\mathrm{dn}(2w_1)}\right)^{1/2}, \tag{31}$$

$$F_1 = (1/\sqrt{2})F((1/\sqrt{2})|\psi_1), \qquad \cos\psi_1 = (R_0^2 - a)/(R_0^2 + a), \qquad R_0^2 \geqslant a;$$

$$\epsilon = 1, \qquad C = 0, \qquad R = \sqrt{2}\,R_0/(R_0\tau + \sqrt{2}); \tag{32}$$

$$\begin{aligned} &\epsilon = +1, \qquad C > 0, \qquad R = (2a)^{1/2}\,\mathrm{dn}(w_2)/\mathrm{sn}(w_2), \\ &F_2 = F((1/\sqrt{2})|\psi_3), \qquad \sin^2\psi_2 = 2a/(R_0^2 + a), \qquad R_0^2 \geqslant a; \end{aligned} \tag{33}$$

$$\begin{aligned} &\epsilon = -1, \qquad C < 0, \qquad R = a^{1/2}\,\mathrm{cn}(w_3), \\ &F_3 = F((1/\sqrt{2})|\psi_3), \qquad \sin^2\psi_3 = (a - R_0^2)/2a, \qquad R_0^2 \leqslant a. \end{aligned} \tag{34}$$

For solutions (31) and (32) H is positive definite. In the other cases H may become negative for $\lambda > 0$.

4. Spherical waves

Another type of solution emerges if we require A to depend on x only through $x_\mu x^\mu$. We use the abbreviations

$$s = |\lambda|\,|x_\mu x^\mu|, \qquad \tau = \ln\sqrt{s}, \qquad \epsilon = \mathrm{sgn}\,\lambda\ \mathrm{sgn}\,x_\mu x^\mu,$$

and write

$$A = (1/\sqrt{s})R(\tau)\,\mathrm{e}^{\mathrm{i}\phi(\tau)}. \tag{36}$$

If we insert this *ansatz* in the field equation (1), we obtain differential equations in τ for R and ϕ. We recover again equation (13) from current conservation. Thus we have equation (14) again and can carry through the same steps as before. The equation for the radial part becomes

$$\ddot{R} - R - \epsilon R^3 - D^2/4R^3 = 0,$$

and we obtain equation (16) again, where now

$$f(x) = \epsilon x^3 + 2x^2 - 2Cx - D^2/2, \tag{37}$$

so that the solution $\rho = R^2$ is again an elliptic function which can be written in one of the forms (I)–(III) given before. The additional term $2x^2$ introduces no new features, but does introduce some computational complications. The current is now

$$j^\mu = (\epsilon D/\lambda)x^\mu/(x_\rho x^\rho)^2, \tag{38}$$

and we have

$$\begin{aligned} H &= \frac{1}{2|\lambda|\,|x_\mu x^\mu|^3}[(x_0^2 + \boldsymbol{x}^2)(2\rho - C - \dot{\rho}) + \epsilon\rho^2\boldsymbol{x}^2], \\ \boldsymbol{P} &= \frac{x_0\boldsymbol{x}}{2|\lambda|\,|x_\mu x^\mu|^3}(2\rho - C - \dot{\rho} + \epsilon\rho^2), \\ L &= \frac{\mathrm{sgn}(x_\mu x^\mu)}{2|\lambda|\,|x_\mu x^\mu|^2}(2\rho - C - \dot{\rho} + \epsilon\rho^2). \end{aligned} \tag{39}$$

We have not found other forms of these quantities from which simple positivity properties can be read off. Neither are the integrals on $|\boldsymbol{x}|$ straightforward. It is observed that the quantities (39) as well as the current (38) decrease rather quickly for large distances x^2 and become singular at the light cone. These properties are related to the factor $s^{-1/2}$ in A, which is a trace of the dilatation invariance of equation (1). With the exception of some degenerate cases (see below), the three types (I)–(III) of spherical solutions are invariant with respect to 'discrete' dilatation transformations: if the logarithm of the scale factor η is proportional to an integer multiple of the period of the solution, A remains unchanged under the corresponding dilatation (the fixed proportionality constant depends on the type). This type of dilatation was discussed some time ago in connection with quantum field theory (Mitter 1964).

We shall now write down the constants appearing in solutions (I)–(III) which are a little more complicated than for plane waves. The integration constants are now

$$D = 2\rho_0\dot{\phi}_0, \qquad C = \rho_0[\tfrac{1}{2}\epsilon\rho_0 - (\dot{\rho}_0/2\rho_0)^2 - \dot{\phi}_0^2 + 1]. \tag{40}$$

With the abbreviation

$$h = 1 + 3\epsilon C/2, \tag{41}$$

we note the following results: Form (I) applies if

$$\begin{aligned} &\text{either } C \leqslant -\tfrac{1}{2} \quad \text{for any } D^2 \geqslant 0, \\ &\text{or } C \geqslant -\tfrac{1}{2} \quad \text{for } 1 + 9C/4 + h^{2/3} \leqslant 27D^2/32, \end{aligned} \tag{42}$$

and we have

$$\begin{aligned} &a_1 = \tfrac{2}{3}[-1 + (\sqrt{\Delta} - 1 - 9C/4 + 27D^2/32)^{1/3} - (\sqrt{\Delta} + 1 + 9C/4 - 27D^2/32)^{1/3}], \\ &\Delta = 27D^4/128 - (D^2/2)(1 + 9C/4) - C^2/2 - C^3, \\ &G = a_1 + \tfrac{2}{3}, \qquad B = (3a_1^2 + 4a_1 - 2C)^{1/2}. \end{aligned} \tag{43}$$

Form (II) or (III) applies if

$$\epsilon C \geqslant -\tfrac{1}{2}, \qquad 1 + 9\epsilon C/4 + h^{2/3} \geqslant 27D^2/32. \tag{44}$$

The roots read

$$\begin{aligned} &a_2 = \tfrac{2}{3}(-\epsilon + 2h^{2/3}\cos\alpha/3) \\ &b_{1,2} = \tfrac{2}{3}[-\epsilon - 2h^{2/3}\cos(\alpha \pm \pi)/3] \\ &\cos\alpha = -(\epsilon/h^{2/3})(1 + 9\epsilon C/4 - 27D^2/32). \end{aligned} \tag{45}$$

For coinciding roots the solutions can, as before, be written in terms of elementary functions. For $\epsilon = 1$, $\Delta = 0$ we obtain

$$\rho = -\tfrac{2}{3} + \tfrac{4}{3}h^{1/2}[1 + \cot^2(\tau h^{1/2} - \beta)], \qquad \sin^2\beta = 2\sqrt{h}/[\rho_0 + \tfrac{2}{3}(1 + \sqrt{h})]. \tag{46}$$

For $\epsilon = -1$, $\Delta = 0$ we have

$$\rho = \rho_0 = \tfrac{2}{3}(1 + \sqrt{h}), \qquad \phi = (D/2\rho_0)\tau. \tag{47}$$

For this solution H is positive. The field only takes up a constant phase factor under (arbitrary) dilatations.

In the general cases we have not found explicit analytic expressions for the phase

$$\phi = \frac{D}{2}\int_0^\tau \frac{\mathrm{d}x}{\rho(x)}.$$

Finally we shall give a complete list of solutions for $D = 0$ (real field). With

$$\gamma = (1+2\epsilon C)^{1/2} \tag{48}$$

we have: for $\epsilon = 1$, $C > 0$,

$$\begin{aligned} &R = (2\gamma)^{1/2}\,\mathrm{dn}(w_1|k_1)/\mathrm{sn}(w_1|k_1), && k_1^2 = (1+\gamma)/2\gamma, \\ &w_1 = \gamma^{1/2}\tau - F(k_1|\psi_1), && \sin^2\psi_1 = 2\gamma/(R_0^2+1+\gamma); \end{aligned} \tag{49}$$

for $\epsilon = 1$, $-\frac{1}{2} < C < 0$,

$$\begin{aligned} &R = (1+\gamma)^{1/2}\,\mathrm{cn}(w_2|k_2)/\mathrm{sn}(w_2|k_2), && k_2^2 = 2\gamma/(1+\gamma), \\ &w_2 = [(1+\gamma)/2]^{1/2}\tau - F(k_2|\psi_2), && \sin^2\psi_2 = (1+\gamma)/(R_0^2+1+\gamma); \end{aligned} \tag{50}$$

for $\epsilon = 1$, $C < -\frac{1}{2}$,

$$\begin{aligned} &R = (-2C)^{1/4}\,\mathrm{cn}(w_3|k_3)/\mathrm{sn}(w_3|k_3)\,\mathrm{dn}(w_3|k_3), && k_3^2 = \tfrac{1}{2}(1-1/\sqrt{-2C}), \\ &w_3 = (-C/2)^{1/4}\tau - \tfrac{1}{2}F(k_3|\psi_3), && \cos\psi_3 = (R_0^2-\sqrt{-2C})/(R_0^2+\sqrt{-2C}); \end{aligned} \tag{51}$$

for $\epsilon = -1$, $C < 0$,

$$\begin{aligned} &R = (1+\gamma)^{1/2}\,\mathrm{cn}(w_4|k_1), \qquad w_4 = \gamma^{1/2}\tau - F(k_1|\psi_4), \\ &\sin^2\psi_4 = (1+\gamma-R_0^2)/(1+\gamma); \end{aligned} \tag{52}$$

for $\epsilon = -1$, $0 < C < \frac{1}{2}$,

$$\begin{aligned} &R = (1+\gamma)^{1/2}\,\mathrm{dn}(w_5|k_2), \qquad w_5 = [(1+\gamma)/2]^{1/2}\tau - F(k_2|\psi_5), \\ &\sin^2\psi_5 = (1+\gamma-R_0^2)/(1+\gamma). \end{aligned} \tag{53}$$

The degenerate cases are: for $\epsilon = 1$, $C = -\frac{1}{2}$,

$$R = (R_0 + \tan\tau/\sqrt{2})/(1 - R_0\tan\tau/\sqrt{2}); \tag{54}$$

for $\epsilon = \pm 1$, $C = 0$,

$$R = R_0/[\cosh\tau - (1 \pm R_0^2/2)^{1/2}\sinh\tau]; \tag{55}$$

for $\epsilon = -1$, $C = \frac{1}{2}$,

$$R = 1. \tag{56}$$

The last solution is invariant under arbitrary dilatations. In addition H is positive.

5. Some applications of spherical solutions

Even with the simple expressions for R given by equations (49)–(56) for real fields it is not always possible to obtain simple formulae for the field

$$A = R(\tau = \ln\sqrt{s})/\sqrt{s}, \qquad s = |\lambda|\,|x_\mu x^\mu|. \tag{57}$$

In general, the structure is dominated by the singularity $s^{-1/2}$ with the superimposition of oscillations which are periodic in τ. For $\epsilon = 1$ these oscillations are themselves unbounded. For $\epsilon = -1$ no additional singularities are introduced. The behaviour for $s \neq 0$ can be studied with the aid of Fourier or power series expansions of the Jacobian functions. Close to $s = 0$ these expansions are not useful, since the oscillations become more and more rapid with decreasing s.

Some of the solutions described here have been used before. If we take R from equation (55) we obtain

$$A = a/(1 - a^2\lambda x^2/8), \tag{58}$$

which exhibits a simple pole for $\lambda x^2 > 0$ and no singularity for the other sign. This singularity has been interpreted as a shock wave (Burt and Pocsik 1976) which moves with a velocity $v \gtreqless c$ for $\lambda \gtreqless 0$ and acquires the speed of light after an infinite time x_0. The solution (58) is also of relevance for the instanton solution of Euclidean gauge field theory (Corrigan and Fairlie 1977).

As has been remarked in § 2 one may obtain from any of the solutions given above new solutions using the invariances of equation (1). This may influence the singularities and change the propagation pattern of the solution. In order to indicate what can happen we shall give an example. If we start from the field (57) with any one of the solutions (49)–(56) and apply a translation $x^\mu \to x^\mu - c^\mu$, followed by a conformal transformation (9), (10) with

$$b^\mu = -c^\mu/2c^2, \tag{59}$$

we obtain a solution in the form

$$A = \frac{K}{[|(x-c)^2(x+c)^2|]^{1/2}} R(\ln\sqrt{\xi}), \qquad \xi = |4c^2\lambda| \left|\frac{(x-c)^2}{(x+c)^2}\right|, \tag{60}$$

with a constant K. A solution of this type with R from equation (53) has been used in connection with gauge theories in Minkowski space (Cervero *et al* 1977) and has provided some insight into the relation of Euclidean instantons and Minkowskian meron solutions. The same procedure applied to (58) gives, with the choice $\lambda a^2 c^2 = -2$, the instanton

$$A = 2ac^2/(x^2 + c^2), \tag{61}$$

whereas (60) with (56) gives (cf de Alfaro *et al* 1976, 1977)

$$A = K/[|(x-c)^2(x+c)^2|]^{1/2}. \tag{62}$$

It is obvious that similar considerations could be carried out for more general transformations than (59), and that they apply also for complex fields and the plane wave solutions of § 3. Whether these considerations are useful depends on the meaning which one wants to ascribe to the field A. Any speculation on this question is beyond the scope of our approach.

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